MMAT5520 Differential Equations & Linear Algebra
Final Exam (29 Nov 2010)
Time allowed: 120 mins

Full marks: 60

1. (8 marks) Consider the equation
$$t^2y'' - 5ty' + 9y = 0,\ t > 0.$$
   (a) The equation has a solution of the form t^k for some integer k. Find k.
   (b) Find the general solution of the equation.

2. (8 marks) Let $L[y] = y'' + y' - 6y$.
   (a) Solve the initial value problem
$$\begin{cases} L[y] = 0 \\ y(0) = 1 \\ y'(0) = 1 \end{cases}.$$
   (b) Use the method of variation of parameter to find a particular solution to the equation $L[y] = e^{-t}$.

3. (8 marks) Let $L[y] = y''' + 4y'$.
   (a) Find a fundamental set of solutions to the homogeneous equation $L[y] = 0$ and show that yours solutions are linearly independent by writing down their Wronskian.
   (b) Write down an appropriate form of a particular solution (do not solve the equation) to the equation
$$L[y] = 2t^2 + e^{3t} - 4t \sin 2t.$$

4. (8 marks) Let $\mathbf{A} = \begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix}$. Find $\exp(\mathbf{A}t)$.

5. (8 marks) Let
$$\mathbf{A} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 4 & 4 \\ 0 & -1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$
   (a) Find a generalized eigenvector of rank 3 of $\mathbf{A}$.
   (b) Solve
   $$\mathbf{x}' = \mathbf{A}\mathbf{x},$$
   where the derivative is taken with respect to t.

6. (10 marks) A stochastic matrix is a square matrix with non-negative entries such that the sum of the entries in each column is one.
   (a) Show that any stochastic matrix has eigenvalue $\lambda = 1$.
   (b) Show that if all entries of a stochastic matrix is positive, then the eigenspace associated with $\lambda = 1$ is one dimensional.
   (c) Let $\mathbf{A} = \begin{pmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{pmatrix}$.
      (i) Diagonalize $\mathbf{A}$.
      (ii) Let $\mathbf{x}_0 = (a, b)^T$, where a, b are real numbers. Find $\lim_{k\to\infty} \mathbf{A}^k\mathbf{x}_0$ in terms of a and b.

7. (10 marks) Let
$$\mathbf{A} = \begin{pmatrix} 2 & 5 \\ -1 & -4 \end{pmatrix}.$$
   (a) Find a matrix function $\Psi(t)$ such that
   $$\frac{d}{dt}\Psi(t) = \mathbf{A}\Psi(t).$$
   (b) Solve the initial value problem
   $$\begin{cases} \mathbf{x}' = \mathbf{A}\mathbf{x} \\ \mathbf{x}(0) = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \end{cases}$$
   where the derivative is taken with respect to t.
   (c) Find a particular solution to
   $$\mathbf{x}' = \mathbf{A}\mathbf{x} + \begin{pmatrix} 3 \\ e^{2t} \end{pmatrix}.$$

Supplementary problems:

8. Let $\mathbf{A}$ be a square matrix with all entries equal to 1. Is $\mathbf{A}$ diagonalizable?

9. Let $\mathbf{P}$ be a square matric with $\mathbf{P}^2 = \mathbf{P}$. Prove that $\mathbf{P}$ is diagonalizable.

10. Let A be an orthogonal matrix. Prove that if λ is an eigenvalue of A, then $|\lambda| = 1$.

11. Let A be a skew-symmetric matrix. Prove that $x^T Ax = 0$ for any vector x.

12. Let A be a skew-symmetric matrix. Prove that $I+A$ is invertible. Solution: Suppose $(I + A)x = 0$. The $0 = x^T(I + A)x = x^T Ix + x^T Ax = x^T x$ Thus $x = 0$.

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